ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION

\((x^2+y^2) - xy + x + y + 1 = 16z^2\)

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ABSTRACT

The ternary homogeneous quadratic equation given by \((x^2 + y^2) - xy + x + y + 1 = 16z^2\) representing a cone is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special polygonal and pyramidal numbers are presented.

INTRODUCTION

The Diophantine equations offer an unlimited field for research due to their variety [1-3]. In particular, one may refer [4-24] for quadratic equations with three unknowns. This communication concerns with yet another interesting equation \((x^2 + y^2) - xy + x + y + 1 = 16z^2\) representing homogeneous quadratic equation with three unknowns for determining its infinitely many non-zero integral points. Also, a few interesting relations among the solutions are presented.

NOTATIONS USED

1. Polygonal number of rank ‘n’ with sides m
   \[ t_{m,n} = n \left(1 + \frac{(n-1)(m-2)}{2}\right) \]

2. Pronic number of rank ‘n’
   \[ Pr_n = n(n+1) \]

METHOD OF ANALYSIS

The ternary quadratic diophantine equation under consideration is

\[(x^2 + y^2) - xy + x + y + 1 = 16z^2 \tag{1}\]

The substitution of the linear transformations

\[ x = u + v \quad ; \quad y = u - v \quad (u \neq 0, v \neq 0) \tag{2}\]

in (1) gives

\[ (u + 1)^2 + 3v^2 = 16z^2 \tag{3}\]

Taking \(u+1=U\)

in (3), it gives

\[ U^2 + 3v^2 = 16z^2 \tag{4}\]

Now, (4) is solved through different methods and thus, different patterns of solutions to (1) are obtained.

Method: 1

Write (4) as

\[ U^2 - 16z^2 = -3v^2 \]

i.e. \((U + 4z)(U - 4z) = -3v^2 \tag{5}\]

Choice (i)

Write (5) in the form of ratio as
This is equivalent to the following two equations

\[
\begin{align*}
-\alpha U - 3\beta v + 4\alpha \gamma &= 0 \\
U\beta - \alpha v + 4\beta \zeta &= 0
\end{align*}
\]  \tag{7}

Applying the method of cross multiplication, the above system of equations is satisfied by

\[
U = u + 1 = 4\alpha^2 - 12\beta^2
\]

\[
v = 8\alpha\beta
\]

\[
z = \alpha^2 + 3\beta^2
\]  \tag{8}

Substituting the values of \(u\) and \(v\) in (2), we get

\[
x = 4\alpha^2 - 12\beta^2 - 1 + 8\alpha\beta
\]

\[
y = 4\alpha^2 - 12\beta^2 - 1 - 8\alpha\beta
\]  \tag{9}

Thus (8) and (9) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

A few interesting properties are as follows:

1. \(x(a, a) - y(a, a) - 16t_{4,a} = 0\)
2. \(x(2a, a) + y(2a, a) - 8t_{4,a} + 2 = 0\)
3. \(z(5b, 5b)\) is a perfect square.
4. \(z(b, 4b)\) is a perfect square.
5. \(x(1, a) + y(1, a) + z(1, a) + 21t_{4,a} - 7 = 0\)

Choice (ii)

(5) is written in the form of ratio as

\[
\frac{-3\nu}{U - 4\zeta} = \frac{(U + 4\zeta)}{v} = \frac{\alpha}{\beta}, \beta \neq 0
\]  \tag{10}

This is equivalent to the following two equations

\[
\begin{align*}
-\alpha U - 3\beta v - 4\alpha \gamma &= 0 \\
\beta U - \alpha v - 4\beta \zeta &= 0
\end{align*}
\]  \tag{11}

Applying the method of cross multiplication, the above system of equations is satisfied by

\[
U = u + 1 = -4\alpha^2 + 12\beta^2
\]

\[
v = -8\alpha\beta
\]

\[
z = \alpha^2 + 3\beta^2
\]  \tag{12}

Substituting the values of \(u\) and \(v\) in (2), we get

\[
x = -4\alpha^2 + 12\beta^2 - 1 - 8\alpha\beta
\]

\[
y = -4\alpha^2 + 12\beta^2 - 1 + 8\alpha\beta
\]  \tag{13}

Thus (12) and (13) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

A few interesting properties are as follows:

1. \(x(2, a) + y(2, a) + z(2, a) - 27t_{4,a} + 30 = 0\)
2. \(x(b, 1) + y(b, 1) + 8t_{4,b} - 22 = 0\)
3. \(z(4b, 4b)\) is a perfect square.
(4) \( y(b,2) - z(b,2) + 2t_{4,b} - 16pr_b - 35 = 0 \).  
(5) \( x(1,a) - z(1,a) - 17t_{4,a} + 8pr_a + 6 = 0 \)

**Choice (iii)**

Also, (5) is written in the form of ratio as

\[
\frac{U + 4z}{-3v} = \frac{v}{U - 4z} = \frac{\alpha}{\beta}, \quad \beta \neq 0
\]  
(14)

This is equivalent to the following two equations

\[
\begin{align*}
U\beta + 3\alpha v + 4\beta z &= 0 \\
-U\alpha + \beta v + 4\alpha z &= 0
\end{align*}
\]  
(15)

Applying the method of cross multiplication, the above system of equations is satisfied by

\[
U= u + 1 = 12\alpha^2 - 4\beta^2
\]

\[
v = -8\alpha\beta
\]

\[
z = 3\alpha^2 + \beta^2
\]  
(16)

Substituting the values of \( u \) and \( v \) in (2), we get

\[
x = 12\alpha^2 - 4\beta^2 - 1 - 8\alpha\beta
\]

\[
y = 12\alpha^2 - 4\beta^2 + 1 + 8\alpha\beta
\]  
(17)

Thus (16) and (17) represent non-zero distinct integral solutions of (1) in two parameters.

**Properties:**

A few interesting properties are as follows:

1. \( x(1,-a) - y(1,-a) - z(1,-a) + 17t_{4,a} - 16pr_a + 3 = 0 \)
2. \( x(2,b) + y(2,b) + z(2,b) + 7t_{4,b} - 106 = 0 \)
3. \( x(2,-b) - z(2,-b) + 2t_{4,b} - 16pr_b - 35 = 0 \)
4. \( x(a,a) + y(a,a) + z(a,a) - 20t_{4,a} + 2 = 0 \)
5. \( x(-a,-a) - y(-a,-a) - z(-a,-a) + 20t_{4,a} = 0 \)

**Choice (iv)**

Again (5) is written in the form of ratio as

\[
\frac{U - 4z}{-3v} = \frac{v}{U + 4z} = \frac{\alpha}{\beta}, \quad \beta \neq 0
\]  
(18)

This is equivalent to the following two equations

\[
\begin{align*}
U\beta + 3\alpha v - 4\beta z &= 0 \\
-U\alpha + \beta v + 4\alpha z &= 0
\end{align*}
\]  
(19)

Applying the method of cross multiplication, the above system of equations is satisfied by

\[
U= u + 1 = -12\alpha^2 + 4\beta^2
\]

\[
v = 8\alpha\beta
\]

\[
z = 3\alpha^2 + \beta^2
\]  
(20)

Substituting the values of \( u \) and \( v \) in (2), we get

\[
x = -12\alpha^2 + 4\beta^2 - 1 + 8\alpha\beta
\]

\[
y = -12\alpha^2 + 4\beta^2 - 1 - 8\alpha\beta
\]  
(21)

Thus (20) and (21) represent non-zero distinct integral solutions of (1) in two parameters.

**Properties:**

A few interesting properties are as follows:

1. \( x(a,3) + y(a,3) + 24t_{4,a} - 70 = 0 \)
Thus (31) (32) and (4) represent non-zero integers.

x = x(a, −a) = a^2 + 3b^2

where a, b are non-zero integers.

Write 16 as

16 = (2 + i2√3)(2 − i2√3)

Substituting (22) and (23) in (4) and employing the method of factorization, we’ve

(2U + i2√3v)(2U − i2√3v) = (2 + i2√3)(2 − i2√3)(a + i√3b)^2(a − i√3b)^2

The above equation is equivalent to the system of equation

(2U + i2√3v) = (2 + i2√3)(a + i√3b)^2

(2U − i2√3v) = (2 − i2√3)(a − i√3b)^2

Equating the real and imaginary parts in (24)

U = u + 1 = u(a, b) = 2a^2 − 6b^2 − 12ab

v = v(a,b) = 2a^2 − 6b^2 + 4ab

Substituting the values of the u and v in (2), we’ve

x = x(a, b) = 4a^2 − 12b^2 − 8ab − 1

y = y(a, b) = −16ab − 1

Thus (26) (27) and (25) represent non-zero distinct integral solutions of (1) in two parameters.

Properties:

1. x(a, 2) − y(a, 2) − 20t_{4,a} + 16pr_a + 48 = 0
2. y(a, 1) + z(a, 1) − 17t_{4,a} + 16pr_a − 2 = 0
3. (8b, 8b) is a perfect square.
4. (12b, 12b) is a perfect square.
5. − x(1, a) + y(1, a) + z(1, a) − 23t_{4,a} + 8pr_a + 3 = 0

Case(ii)

Write 16 as

16 = (−2 + i2√3)(−2 − i2√3)

Substituting (22) (28) in (4) and employing the method of factorization, we’ve

(−2U + i2√3v)(−2U − i2√3v) = (−2 + i2√3)(−2 − i2√3)(a + i√3b)^2(a − i√3b)^2

The above equation is equivalent to the system of equation

(−2U + i2√3v) = (−2 + i2√3)(a + i√3b)^2

(−2U − i2√3v) = (−2 − i2√3)(a − i√3b)^2

Equating the real and imaginary parts in (29)

U = u + 1 = u(a, b) = −2a^2 + 6b^2 − 12ab

v = v(a, b) = 2a^2 − 6b^2 − 4ab

Substituting the values of the u and v in (2), we’ve

x = x(a, b) = −16ab − 1

y = y(a, b) = −4a^2 + 12b^2 − 8ab − 1

Thus (31) (32) and (4) represent non-zero distinct integral solutions of (1) in two parameters.
Properties:-
(1) \( x(-a,-a) + y(-a,-a) + 16t_{4,a} + 2 = 0 \)
(2) \( x(1,a) + z(1,a) - 19t_{4,a} + 16pr_a = 0 \)
(3) \((32b,32b)\) is a perfect square.
(4) \((24b,24b)\) is a perfect square.
(5) \( x(1,a) + y(1,a) + z(1,a) + 39t_{4,a} + 24pr_a + 5 = 0 \)

CONCLUSION
In this paper, we have obtained infinitely many non-zero distinct integer solutions to the ternary quadratic diophantine equation represented by \((x^2 + y^2) - xy + x + y + 1 = 16z^2\). As quadratic equations are rich in variety, one may search for other choices of quadratic equation with variables greater than or equal to 3 and determine their properties through special numbers.

REFERENCES


